Abstract: This research incorporates the use of Microsoft Excel to perform engineering design with focus on predicting performance parameters of water rockets.

The study draws on a previous research endeavor that validated the outputs of the water rocket design spreadsheet used in this lab activity with experimental data to a confidence interval. The spreadsheet was improved to return even more accurate predictions of water rocket flight performance, even within the context of a highly variable student lab environment. Because the spreadsheet was created for use in the student lab environment, the test conditions for validation of the spreadsheet were matched as close as possible to those of the student lab environment.

This paper outlines a recommended lab exercise for the university classroom setting that involves student’s designing a water rocket using optimization tools and basic materials such as soda bottles and cardboard. The students then build their rocket, predict the maximum height their rocket will achieve and the impulse imparted to their rocket, and finally, experimentally test their rocket and compare their predictions to their experimental data.

Module Objectives:
Upon completion of this activity, students will be able to:
1. Design, optimize, and build a water rocket for maximum performance, and predict the water rocket’s performance.
2. Test the water rocket’s performance in its operating environment, and use software and analytical tools to analyze experimental data obtained from testing.
3. Critically evaluate experimental performance of their rocket with reference to their predictions, testing methods, and relevant literature data.
4. Calculate the impulse imparted to their water rocket using a recorded flight duration and the empty mass of their rocket.

MatEdu Core Competencies Covered
0.A. Demonstrate Good Communication Skills
0.B. Prepare tests and analyze data
1.A. Carry out measurements of dimensions and of physical phenomena
1.C. Demonstrate Laboratory Skills
2.A. Apply Basic Mathematics Fundamentals
2.C. Apply Geometry and Trigonometric Functions
2.E. Demonstrate Appropriate Use of Statistics
3.B. Demonstrate use of computer applications
4.A. Demonstrate Effective Work with Teams
6.A. Apply Basic Concepts of Mechanics
7.J. Demonstrate how materials properties are used in engineering design

Key Words:
Water Rockets, Spreadsheet Optimization, Impulse, Dynamics

Type of Module/Mode of Presentation:
This activity includes in-class, demo, and lab aspects.

Time Required:
This is a lab activity that requires 2-3 weeks duration, 2-3 lab interactions 2 hours in length depending on curricula and infrastructure for launching water rockets.

Prerequisite Knowledge:
1. Basic logic and math skills (at least pre-calculus; integral calculus if covering numerical methods).
2. Use of spreadsheets to perform calculations using formulas.
3. Knowledge of force vectors, basic kinematics, and impulse.

Target Grade Level:
This activity is oriented to Grades 11-16 (Undergraduate College including running start students).

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Equipment and Supplies Needed:
1. Design/Build and Optimization
a. Computer with Excel®
b. Smart spreadsheet
c. Calipers
d. Knowledge of kinematics, fluid mechanics, and impulse/energy relations
e. Basic knowledge of fin and rocket ballast functionality
f. Knowledge of water and air properties
g. Knowledge of soda bottle volume
h. Soda bottles (any size)
i. Heavy duty tape
j. Cardboard/other thick natural material
k. Scissors
l. Sand/dirt/small rocks
m. Scale (at least 1 kg)

2. Testing and Data Analysis
a. Water rocket launchers (2 is recommended)
b. Graduated cylinders (1 L)
c. 10 G water
d. 120 fps minimum cameras [you may use stop watches instead but this will lead to inaccurate experimental data]
e. Basic video editing software [must allow you to look at each frame of video footage individually]

Curriculum Overview and Instructor Notes:
The instructor may wish to modify any of the elements of this work based on curricula requirements or infrastructure restrictions. This includes the design tool (making improvements to the spreadsheet), the materials processing used to construct the water rockets (the use of composites in construction [see Johnson, 2008] and/or 3D printed components such as nozzles, machined components such as precise ballast weights to eliminate change in mass due to successive rocket launches), and evaluation (testing methods and equipment for measuring a maximum height and impulse).

There are hundreds of different water rocket activities in the literature but not many that specifically target undergraduate level students or utilize ‘smart spreadsheets’, so this module specifically addresses the need for a water rocket simulator spreadsheet that not only produces accurate predictions, but is also housed in an open platform such as Excel® that will allow undergraduate students to explore how the spreadsheet works and how each input variable impacts the final predictions. It also allows educators to make changes to the simulator as they see fit to match their curricula requirements or make improvements to the spreadsheet. All in all, students will design and optimize a water rocket, build the rocket, and then test the rocket to evaluate experimental results as they compare to predictions and variations in testing methods.

This lab is traditionally done within the context of a ‘Dynamics’ course lab section to secure student understanding of fundamental impulse principles and allow for a little fun in the spring time. This module adds an additional component of using smart spreadsheets to quickly design and optimize for performance as well as analyze experimental data, involving multiple
disciplines of engineering. The lab activity is also easily transferable to an ‘Introduction to Engineering’ or a ‘Fluid Mechanics’ course\textsuperscript{4,5}. Students are often exposed to the concepts of these courses and may perform rudimentary analyses using Excel\textsuperscript{®}, but seldom are exposed to the application of these principles or the more extensive tasks they could accomplish with Excel\textsuperscript{®}. This lab exercise accomplishes both. Because the spreadsheet includes a complete walkthrough of how the simulator works and is simply a downloadable Excel\textsuperscript{®} file, it allows students to explore at their own pace, matched with their own desire to learn.

Because water rockets entail a variable flow rate as well as a variable mass of the rocket, the calculations are much more complicated than a real design example. The derivation of a height and impulse involves a fluid mechanics component and a kinematics component. Depending on your curricula requirements, you can focus on either of these. This module focuses on the kinematics portion. There reaches a point on the kinematics side of the derivations where numerical integration is inevitable, but for the purposes of your specific curricula, the spreadsheet allows for this portion to be skipped all together.

The lab also attempts to find a middle ground for effective testing methods for water rocket performance. The use of stop watches to measure rocket flight time is extremely inaccurate and leads to large gaps between predictions and experimental results. Bypassing the need for a flight duration by using altimeters is expensive, requiring that you also use a parachute with the water rockets to protect the altimeter. This module uses cameras that record footage 120 frames per second to acquire a flight duration (120 fps results in a precision the same as standard rocket altimeters). The launch and landing is recorded, the footage is then taken to video editing software to look at the footage frame by frame. The launch and landing frames are identified. Finally, the flight duration obtained is entered into the spreadsheet at ‘Step 5’ to acquire a maximum height (by numerical methods). This method has been verified by a previous study\textsuperscript{1}.

**Module Procedure:**

*Note: Please observe lab safety policies during this activity.*

**Objective:**
The objective of this lab is to design and optimize the parameters of a water rocket using a smart spreadsheet, predict the maximum height and accumulated impulse of the rocket, build and test the rocket, and evaluate the experimental data collected with respect to the initial predictions and testing methods.

**Scope:**
Teams of 2-3

Week 1 – Design/optimize/build water rocket using smart spreadsheet. Turn in spreadsheet predictions at the beginning of lab next week.

Week 2 – Launch rockets and record data using cameras. Begin lab report.
Week 3 – Analyze raw data from rocket launches. Individual lab report due at the beginning of lab next week.

Tasks:
1. Choose a soda bottle volume to use for the base of your rocket.
2. Build your rocket to have fins, a nose cone, and ballast chamber using the supplies provided.
3. Enter the ‘Step 1’ parameters of your rocket into the smart spreadsheet (this is the geometric parameters of your rocket acquired by using a set of calipers and scale).
4. Move on to ‘Step 2’ in the smart spreadsheet: note your optimized ballast weight and enter in your chosen ballast weight into the appropriate cell. Then weigh out this mass in sand/dirt and place it into the ballast chamber of your rocket.
5. Move on to ‘Step 3’ of the smart spreadsheet: We will use 60 psi as our initial pressure. Experiment with the smart spreadsheet to see how it assists in determining an initial pressure that results in safe stress states for the soda bottle material used in your rocket design.
6. Move on to ‘Step 4’ of the smart spreadsheet: decide on a water volume to use at launch for your water rocket. Note the optimized water volume and view the graph of work/unit mass vs. water volume on the next sheet of the spreadsheet. Enter the other information into the user input cells of ‘Step 4’ such as the estimated temperature for the day of launch, etc.
7. Launch your water rocket at least 5 times at the initial parameters you designed for using the provided water rocket launcher while recording the launch and landing of each flight with the provided cameras. The more data you collect using the same independent variables, the more statistical power your results will have.
8. Determine the flight durations of each flight using a movie editing software of your choice (hint: www.shotcut.org).
9. Enter each of your determined flight durations into ‘Step 5’ of the smart spreadsheet to acquire a maximum height attained for each flight and enter your data in the designated fields. Observe the statistical analysis of your data and use this to compose a lab report due in two weeks.
10. Also calculate the impulse imparted to your rocket during each flight using your maximum height data and kinematic relations.
11. Include check calculations for any hand analyses performed.

Lab Report Requirements:
— Evaluate your experimental results in terms of your predictions (use % error) and explain any differences between your experimental and predicted results in terms of any variations in your testing methods and factors not accounted for by the smart spreadsheet (this is a discussion of your results).
— Include a copy of your spreadsheet predictions and experimental data in the appendix of your report.
— Abide by the MET Lab Report Guidelines (Introduction, Procedure, Results, Discussion, Conclusion, and Appendix).

The smart spreadsheet is attached and is fully self-explanatory.
Author Comments:

The spreadsheet a part of this lab activity was designed for activities such as this one directed at undergraduate students. Though it is intended for instructors to make modifications and improvements to the attached spreadsheet, it is not intended for instructors to use a different program. All current simulator products for water rocket activities on the market have been shown to be ineffective for the desired learning outcomes discussed in this module\(^1\).

Even though the spreadsheet is very self-explanatory, it will require an introduction in the classroom. It will also need to be emphasized to the students that they can optimize their ballast mass of their water rocket using the smart spreadsheet. The spreadsheet is made available to the students using Canvas™.

The lab was test ran with two weeks allotted to the lab and with the use of altimeters. The lab described here reflects the required changes recognized by the evaluations of multiple observing instructors, but has not yet been demonstrated in the classroom by the author. However, the lab has been performed by the author individually.

The construction of the water rockets is intended to be shallow, and allow students to be creative with fin design, length of the rocket, nose cone shape, etc. The components of construction that most significantly impact flight performance are the weight of the rocket and the weight of the ballast. If fins follow the typical fin shape, size, orientation, and position of benchmark fin designs, they will nearly always ensure stable flight of the rocket leading to accurate spreadsheet predictions.

Using cameras to record rocket launch and landing is the most consistent way to get accurate flight data. The video editing software is required because some cameras don’t allow you to view footage frame by frame on the camera device and video players such as Windows Media Player® typically have a frame resolution built into the program. The software noted in the procedure above matches frame resolution of the footage up to 1000fps. 120fps cameras are recommended because they provide the same precision as a standard model rocket altimeter (granted the numerical integration of the data is correct). This method has its disadvantages in that the maximum height is derived rather than measured directly using an altimeter, but this method has shown to give the same accuracy of results as an altimeter would.

Milestones are set for the student throughout the duration of the lab activity, but it is not necessary to grade each milestone individually. The milestone can simply be checked off by the instructor as complete to ensure the students are staying on track.

The students are evaluated in two ways: 1) their ability to use smart spreadsheets to accomplish design and analysis goals and 2) their ability to evaluate experimental data with respect to theory, performance predictions, and testing environment. Both abilities demonstrated by the students are evaluated in their lab report at the end of the lab activity. The report shows their predictions and raw data, and holds their discussion of their results. Other evaluation points
available in this lab activity are in communication, continual learning, statistical analysis of data, consideration of material properties in engineering design, and possibly integration of engineering principles from different courses. The students are evaluated as teams (during the lab activity) and individually (through their lab report).

Enthusiasm of the students during this lab activity is more than evident in their faces that light up at the thought of blasting things into the air. This enthusiasm assists the instructor in being able to drive home important concepts in the curricula, whether it be impulse or Bernoulli’s equation, which actually drive home. Although water rockets aren’t a realistic design scenario, they provide an environment of fun learning for the students and remind students of the fun they look forward to when they start their careers. They won’t be designing water rockets, but they will be partaking in engineering work that has a similar effect.

Outcomes associated with this lab activity have yet to be linked to ABET outcomes and metrics, but this is planned to be a part of future development.

**Supporting Materials**
See attached spreadsheet and appendices (Calculation Verification and List of Equations)

**References**

**Acknowledgement**
The water rocket lab activity was initially created by Roger Beardsley, Central Washington University.

**Evaluation Packet**
Student Evaluation Questions (to be included in lab report):
2. Explain how water volume affects the performance of the water rocket. Use the graphs from the smart spreadsheet to aid your discussion.
3. How did the material properties of the water rocket bottle influence the design parameters of your water rocket?
4. Why was having a ballast and fins important for the performance of the water rocket?
5. What was the average percent error of your experimental results and what limits of variation would you place on it? Explain.
6. Describe the change in impulse of the water rocket and identify the different stages of the water rocket’s flight using only the impulse over time graph in the smart spreadsheet.

Instructor Evaluation Questions (for the instructor):
1. Did the activity work as presented?
2. Was the background material on fluid mechanics and dynamics sufficient for your background and discussion with the students?
3. Did the activity generate interest among the students?
4. Were the data acquisition methods appropriate for your curricula and infrastructure?
   What would be a better way to collect data?
5. Were the students able to navigate through the smart spreadsheet easily or did it require a lot of extra individual guidance by the instructor?
6. Did the predictions line up with the experimental results? If not, why do you think they did not?
7. Please provide other comments or suggestions for this activity.

Course Evaluation Questions (for the students):
1. Was the activity clear and understandable?
2. Was the instructor’s explanation comprehensive and thorough?
3. Was the instructor interested in your questions?
4. Was the instructor able to answer your questions?
5. Was the importance of material properties in engineering design made clear?
6. Was the importance of kinematics and impulse in engineering design made clear?
7. Did access to the smart spreadsheet deepen your understanding of the concepts presented? Would you use the smart spreadsheet you were given to develop your own smart spreadsheet in the future?
**Given:**
- $D_{rz}$, $D_{bot}$, $L_t$, $L_p$, $m_e$, $t_{wall}$, $V_{tor}$, $P_w$, $k$, $C_a$, $M_b$
- $P_o$, $\sigma_y$, $\sigma_s$, $E$, $e$, $L_p$, $V_w$, $M_a$, $\Theta_0$, $T$, $g$, $P_{air}$, $R_{air}$, $D_t$

**Find:**
- $M_{bo}$, $P_{max}$ [$\sigma_y$], $P_{max}$ [$\sigma_s$], Ballast Sufficiency
- wall type [Pressure vessel], $\sigma_r$, $\sigma_m$, $\sigma_a$, $V_{act}$, $V_{wo}$, $U$, $W$
- $I$, $h_{exp}$, % Error, $h_{exp \ max}$, $S_h$, $S_{exp}$, % Error, $S_{\% \ err}$
- $S_{\% \ err}$, $C_l$, % Error

**Assume:**
- $P_w$ & $P_{air}$ = constant
- $M_o$ = constant
- Numerical Integration can be used to accurately calculate $h_{exp \ max}$ & $C_D$ experimental
- $\sigma_y$, $\sigma_s$, & $E$ are generalizable for all PET soda bottles
- Water storage chamber can be treated as a pressure vessel

**Overview:**
1) BALLAST CALCULATIONS
2) PRESSURE/STRESS/ACTUAL VOLUME CALCULATIONS
3) DRAUGHT COEFFICIENT [EXPERIMENTAL] CALCULATIONS
4) FLIGHT PERFORMANCE PREDICTION CALCULATIONS
5) EXPERIMENTAL DATA ANALYSIS CALCULATIONS

**Notes:**
- See 'List of Equations' for legend
**Ballast Sufficiency**

If \( C_m \geq 1.125C_p \), 'GO'

\[
C_m = \frac{\frac{1}{4} L_T M_e + (L_T - 0.006 L_B) M_B}{M_e + M_B}
\]

\( C_p = \frac{1}{2} L_T \)

**Check Calculations**

\[
C_m = \frac{\frac{1}{4}(4.191m)(10807kg) + (4.191m - 0.006(10953m)) \times (1089kg)}{[1089kg + 10807kg]} = 0.237m
\]

\( C_p = \frac{1}{2}(4.191m) = 0.20955m \)

\( 0.237m > 1.125(0.20955m) \)

\( 0.237m \geq 0.234m \) 'GO'

**Optimized Ballast Mass**

Optimum condition = \( C_m = 1.125C_p \)

\[
\frac{\frac{1}{4} L_T M_e + (L_T - 0.006 L_B) M_B}{M_e + M_B} = 1.125(\frac{1}{2} L_T)
\]

\[
\frac{\frac{1}{4} L_T M_e + (L_T - 1.060 L_B) M_B}{M_e + M_B} = 1.025L_T(M_e + M_B)
\]

**Notes**

Assume center of mass of rocket w/o ballast mass is \( \frac{1}{4} \) length of rocket due to fins mounted at tail of rocket.

Assume center of pressure of rocket is at the center of the rocket.

1/25 is a safety factor tested in the field.
\[ \text{Ballast Calculations} \]

1. \[ 0.25 \text{Lt}_{m_e} + \text{Lt}_{MB} - 1.6666 \text{LB}_{MB} = 1.5625 \text{Lt}_{m_e} + 1.5625 \text{Lt}_{MB} \]

2. \[ \text{Lt}_{MB} - 1.6666 \text{LB}_{MB} = 1.3125 \text{Lt}_{m_e} + 1.5625 \text{Lt}_{MB} \]

3. \[ \text{Lt}_{MB} - 1.6666 \text{LB}_{MB} - 1.5625 \text{Lt}_{MB} = 1.3125 \text{Lt}_{m_e} \]

4. \[ 1.4375 \text{Lt}_{MB} - 1.6666 \text{LB}_{MB} = 1.3125 \text{Lt}_{m_e} \]

5. \[ m_B \left( 1.4375 \text{Lt} - 1.6666 \text{LB} \right) = 1.3125 \text{Lt}_{m_e} \]

6. \[ m_B = \frac{1.3125 \text{Lt}_{m_e}}{1.4375 \text{Lt} - 1.6666 \text{LB}} \]

7. **Check Calculations**

   \[ m_B = \frac{1.3125 \left( 1.4191 \text{m} \right) \left( 10807 \text{ kg} \right)}{1.4375 \left( 1.4191 \text{m} \right) - 1.6666 \left( 10953 \text{ m} \right)} = 10.88 \text{ kg} = 88 \text{ g} \]
**WALL TYPE**

\[ r_0 = \frac{OD_{bot}}{2}, \quad r_{in} = \frac{OD_{bot}}{2} - t_{wall} \]

**THIN WALL:** \( t_{wall} \leq \frac{1}{11} r_{in} \)

**STRESSES:**

**THIN WALL:** \( t_{wall} = 1000254 \text{ m} \)

\[ \sigma_r = P_0 \frac{r_{in} + r_0}{2} / t_{wall} \]

\[ \sigma_h = P_0 \frac{r_{in} + r_0}{2} / 2 t_{wall} \]

**CHECK CASES:**

\[ r_0 = 1.00398 \text{ m} / 2 = 0.50199 \text{ m} \]

\[ r_{in} = 1.00399 - 1.000254 = 0.00374 \text{ m} \]

\[ 1000254 \text{ m} \geq 1.1 (0.04674 \text{ m}) \]

\[ 1000254 \text{ m} \geq 10004074 \text{ m} \]

\[ \sigma_r = 41 \text{ psi} \left( \frac{6894.75 \text{ Pa/psi}}{1000254 \text{ m}} \right) = 2820085 \text{ Pa} \]

\[ \sigma_h = \left( \frac{2820085 \text{ Pa}}{1000254 \text{ m}} \right) \left( \frac{0.04674 \text{ m} + 0.04699 \text{ m}}{2} \right) / 1000254 \text{ m} \]

\[ = 52155392 \text{ Pa} \]

\[ \sigma_a = \left( \frac{2820085 \text{ Pa}}{1000254 \text{ m}} \right) \left( \frac{0.04674 \text{ m} + 0.04699 \text{ m}}{2} \right) / 2 (1000254 \text{ m}) \]

\[ = 2604770910 \text{ Pa} \]

**NOTES:**

\( r' \) is the midplane of the water rocket cylindrical wall in terms of radius. This assumes that the tensile force acts at the centroid of the cross-sectional area, which simulates the force acting on the entire cross-section; this is closer to reality than using \( r_{in} \).

\( P_0 = \text{Gauge Pressure} \)
STRESSES

THICK WALL

\[ t_{\text{wall}} = 1.005 \text{ m} \]

\[ \sigma_r = P \left( \frac{r_0^2 + r_1^2}{r_0^2 - r_1^2} \right) \]

\[ \sigma_h = P \left( \frac{r_1^2}{r_0^2 - r_1^2} \right) \]

\[ \sigma_A = P \left( \frac{r_1^2}{r_0^2 - r_1^2} \right) \]

CHECK CASES

\[ r_1 = 1.04199 \text{ m} - 1.005 \text{ m} = 0.0369 \text{ m} \]

\[ 1.005 \text{ m} \geq 1.1 (0.04199 \text{ m}) \]

\[ 1.005 \text{ m} \geq 1.004199 \text{ m} \]

\[ \begin{array}{l}
\sigma_r = \text{SEE PAGE 4} \\
\sigma_h = (2,826.085 \text{ Pa}) \left( \frac{1.04199 \text{ m}^2 + 0.04199 \text{ m}^2}{1.04199 \text{ m}^2 - 0.04199 \text{ m}^2} \right) = 2,523,274 \text{ Pa} \\
\sigma_A = (2,826.085 \text{ Pa}) \left( \frac{0.04199 \text{ m}^2}{1.04199 \text{ m}^2 - 0.04199 \text{ m}^2} \right) = 112,029 \text{ Pa} \\
\end{array} \]
\[ \varepsilon_v = \frac{\Delta V}{V_0} = \frac{V_{\text{ACT}} - V_0}{V_0} \]

\[ \frac{\Delta V}{V_0} = (1 + \varepsilon_1)(1 + \varepsilon_2)(1 + \varepsilon_3) - 1 \quad \text{[KELLY, 2015]} \]

* For cylindrical pressure vessels, \( \varepsilon_1 = \varepsilon_2 \)

\[ \frac{\Delta V}{V_0} = (1 + \varepsilon_1)(1 + \varepsilon_1)(1 + \varepsilon_3) - 1 \]

\[ = (1 + \varepsilon_1 + \varepsilon_1 + \varepsilon_1^2)(1 + \varepsilon_3) - 1 \]

\[ = (\varepsilon_1^2 + 2\varepsilon_1 + 1)(1 + \varepsilon_3) - 1 \]

\[ = \varepsilon_1^2 + 2\varepsilon_1 + \varepsilon_3 + \varepsilon_1^2\varepsilon_3 + 2\varepsilon_1\varepsilon_3 + \varepsilon_3 - 1 \]

\[ = \varepsilon_1^2 + 2\varepsilon_1 + \varepsilon_3 + \varepsilon_1^2\varepsilon_3 + 2\varepsilon_1\varepsilon_3 + \varepsilon_3 - \varepsilon_1 \]

\[ = \left( \frac{\sigma_{\text{m}}}{E} \right)^2 + \left( \frac{\sigma_A}{E} \right)^2 + 2 \left( \frac{\sigma_{\text{m}}}{E} \right) \left( \frac{\sigma_A}{E} \right) + 2 \left( \frac{\sigma_{\text{m}}}{E} \right) + \left( \frac{\sigma_A}{E} \right) \]

\[ = \frac{\sigma_{\text{m}}^2}{E^2} + \frac{\sigma_A^2}{E^2} + \frac{2\sigma_{\text{m}}\sigma_A}{E} + \frac{2\sigma_{\text{m}}}{E} + \frac{\sigma_A}{E} \]

\[ = \frac{\sigma_{\text{m}}^2}{E^2} + \frac{\sigma_A^2}{E^2} + \frac{2\sigma_{\text{m}}\sigma_A}{E} + 2\sigma_{\text{m}} + \sigma_A \]

\[ V_{\text{ACT}} = \left( \frac{\sigma_{\text{m}}^2}{E^2} + \frac{\sigma_A^2}{E^2} + \frac{2\sigma_{\text{m}}\sigma_A}{E} + 2\sigma_{\text{m}} + \sigma_A \right) V_0 + V_0 \]

NOTES:

* Small strain is not assumed.
* Only true for elastic region.
\[ V_{act} = \left[ \left( \frac{52155392 \text{ Pa}}{2.1088955 \times 10^6 \text{ Pa}} \right)^2 + \frac{2(52155392 \text{ Pa})(2.10877696 \text{ Pa})}{2.1088955 \times 10^6 \text{ Pa}} + \frac{2(52155392 \text{ Pa}) + (2.10877696 \text{ Pa})}{2.1088955 \times 10^6 \text{ Pa}} \right] (1030 \text{ mL}) + (1030 \text{ mL}) \\
= (3.1649 \times 10^{-6} + 7.52422 \times 10^{-4} + 4.8490391 \times 10^{-2}) (1030 \text{ mL}) + (1030 \text{ mL}) \\
= 1081 \text{ mL} \]

*Assuming small strain,
\[ V_{act} = \left( \frac{2(52155392 \text{ Pa}) + 2.10877696 \text{ Pa}}{2.1088955 \times 10^6 \text{ Pa}} \right) (1030 \text{ mL}) + (1030 \text{ mL}) \\
= 1080 \text{ mL} \]

**Notes:**
There is evidence that small strain can be assumed. However, these equations will not be continued in the spreadsheet.
OPTIMUM PRESSURE FOR STAYING BELOW YIELD:

\[ \sigma_R = P_g \]
\[ \sigma_h = P_g \left( \frac{R_{in} + R_{o}}{2} \right) \cdot \frac{t_{wall}}{} \]
\[ = P_g \left( \frac{R_{in}^2 + R_{o}^2}{R_{o}^2 - R_{in}^2} \right) \text{ [THICK WALL]} \]

*SUBSTITUTE \( \sigma_y \) FOR \( \sigma_R \) & \( \sigma_h \) ' SF = 1.05

\[ P_g = \sigma_y \text{ OR } \left( \frac{195 \sigma_y \cdot t_{wall}}{R_{in} + R_{o}} \right) \text{ OR } \sigma_y \left( \frac{R_{in}^2 + R_{o}^2}{R_{o}^2 - R_{in}^2} \right) \]

*APPLY SF

\[ P_g = \text{MINIMUM} \left( 0.95 \sigma_y, \left( \frac{195 \sigma_y \cdot t_{wall}}{R_{in} + R_{o}} \right), 0.95 \sigma_y \left( \frac{R_{in}^2 + R_{o}^2}{R_{o}^2 - R_{in}^2} \right) \right) \]

CHECK CASES:

\[ P_g = 1.95 \left( 92.5 \times 10^6 \text{ Pa} \right) = 87,875 \times 10^6 \text{ Pa} \left( \frac{1 \text{ Psi}}{6894.75 \text{ Pa}} \right) = 12.745 \text{ psi} \text{ OR } \]

\[ = 1.95 \left( 92.5 \times 10^6 \text{ Pa} \right) \left( \frac{0.000254 \text{ m}}{109.674 \text{ m} + 0.04699 \text{ m}} \right) = 476,267 \text{ Pa} \left( \frac{1 \text{ Psi}}{6894.75 \text{ Pa}} \right) = \text{69 psi } \text{ OR } \]

\[ = 1.95 \left( 92.5 \times 10^6 \text{ Pa} \right) \left( \frac{0.04674 \text{ m}^2 + 0.04699 \text{ m}^2}{0.04699 \text{ m}^2 - 0.04674 \text{ m}^2} \right) = 1,185 \times 10^6 \text{ Pa} \left( \frac{1 \text{ Psi}}{6894.75 \text{ Pa}} \right) = 2340 \text{ ksi} \]

NOTES:

THE HOOP STRESS WILL ALWAYS BE HIGHER THAN THE AXIAL STRESS. THE RAPID STRESS MAY BE HIGHER THAN THE HOOP STRESS IN THICK WALL SCENARIOS SO IT WAS CONSIDERED IN CALCULATION.

1.05 SAFETY FACTOR WAS USED BASED ON EXPERIMENTAL DATA FOR YIELD & LMTS OF PET BODY BOTTLES.

CHOOSE THE MIN PRESSURE VALUE w/ RESPECT TO WALL TYPE (THIN OR THICK).
OPTIMUM PRESSURE FOR AVOIDING FAILURE:

\[ P_j = \text{MINIMUM} \left( 0.95 \sigma_{uts}, \frac{1.95 \sigma_{uts \text{ wall}}}{\left( \frac{V_{in} + V_o}{2} \right)}, \frac{0.95 \sigma_{uts}}{\left( \frac{V_{in}^2 + V_o^2}{V_o^2 - V_{in}^2} \right)} \right) \]

CHECK CALC.:

\[ P_j = 0.95 \left( 10000 \, \text{Pa} \right) \left( 170 \, \text{ksi} \right) / \left( \left( 1.04 \left( 0.74 \, \text{m} + 0.4699 \, \text{m} \right) / 2 \right) \right) \]

\[ = 875301 \, \text{Pa} \left( \frac{1 \, \text{Psi}}{6894.75729 \, \text{Pa}} \right) = 127 \, \text{PSI} \]
Height of Rocket

\[ \sum F_y = ma \rightarrow a = \frac{\Sigma F_y}{m} \]

\[ a = \frac{-F_D - W}{m} = -\frac{F_D}{m} - \frac{W}{m} = -\frac{F_D}{m} - g \]

\[ F_D = \frac{1}{2} \rho_{air} A_{bot} u^2 = \frac{1}{2} \rho_{air} (4 \pi OD_{bot}^2) lu u C_D \]

\[ m = m_s \]

\[ a = \frac{-0.5 \rho_{air} \pi OD_{bot}^2 lu u C_D + g}{m_s} = \frac{du}{dt} \]

\[ u = \int_{t_0}^{t} du = \int_{t_0}^{t} \left( \frac{-0.5 \rho_{air} \pi OD_{bot}^2 lu u C_D + g}{m_s} \right) dt \]

* APPROXIMATE W/ A EULER NUMERICAL INTEGRATION

\[ u(t + \Delta t) \approx u(t) + \left( \frac{-0.5 \rho_{air} \pi OD_{bot}^2 lu(t) u C_D + g}{m_s} \right) \Delta t \]

* APPROXIMATE HEIGHT W/ A SIMPSON NUMERICAL INTEGRATION [HIGHER ACCURACY]

\[ h = \int_{t_0}^{t} dh = \int_{t_0}^{t} u \ dt \]

\[ h(t + \Delta t) \approx h(t) + \frac{\Delta t}{\Delta t} (u(t)) + \frac{1}{2} \left( u(t) + u(t+\Delta t) \right) + u(t+\Delta t)) \]

Notes:

To determine the experimental drag coefficient, the rocket is dropped from a known height & the fall time is measured w/ a minimum precision of 0.001 s. A numerical integration is used to calculate height of rocket over time. The fall time is looked up in the numerical integration table & the respective height is output, plug in \( C_D \) until output height matches known drop height.

\( u(t) \) is a previous value while \( u(t + \Delta t) \) is the instantaneous value. \( \Delta t \) is the incremental change in time; increments are added down a column from \( t_0 \) to \( t \).
CHECK CAUSES

\[ u(t+\Delta t) \approx (0) + \left( -0.5 \cdot (1.2 \text{ kg/m}^3) \cdot 0.25 \pi \cdot (1.00398 \text{ m})^2 \cdot (0) \cdot (0) \cdot C_0 \right) \left( \frac{1}{(181.9 \text{ g}) \cdot (1.3 \text{ kg})} \right) + 9.807 \frac{\text{m/s}^2}{\text{m/s}^2} \]

\[ \approx (0 + 9.807 \text{ m/s}^2) \cdot (0.01 \text{ s}) \]

\[ \approx 0.1009807 \text{ m/s} \]

\[ u(t+\Delta t) \approx 1.009807 \text{ m/s} \]

\[ h(t+\Delta t) \approx (0) + \left( \frac{1.0013}{6} \right) \left( 0 + 4 \left( \frac{0 + 0.009807 \text{ m/s}}{2} \right) + 1.009807 \text{ m/s} \right) \]

\[ \approx (0.000121 \text{ s}) \cdot (102.9421 \text{ m/s}) \approx 4.904 \times 10^{-2} \text{ m} \]

\[ h(t+\Delta t) \approx 4.904 \times 10^{-2} \text{ m} \]

**NOTES:**

The rocket fell \( 4.904 \times 10^{-2} \text{ m} \) from \( t = 0 \text{ s} \) to \( t = 0.01 \text{ s} \).
OPTIMUM WATER VOLUME

\[ W = \int_{V_0}^{V_5} P \left( \frac{V}{V_0} \right)^k \, dV \]

subject to \( PV^k = \text{constant} \) \( \Rightarrow P_i V_0^k = \text{constant} \)

\[ \dot{W} = \int_{V_3}^{V_5} P \left( \frac{V}{V_3} \right)^k \, dV \]

\[ W = P_0 V_3^k \int_{V_3}^{V_5} \left( \frac{V}{V_3} \right)^{-k+1} \, dV \]

\[ = P_0 V_3^k \left[ \frac{V_5^{-k+1}}{-k+1} \right] - P_0 V_3^k \left[ \frac{V_3^{-k+1}}{-k+1} \right] \]

\[ = -\frac{P_0 V_3^k}{-k+1} \left( V_5^{-k+1} - V_3^{-k+1} \right) \]

\[ V_0 = V_3 - V_w \quad ; \quad f = \frac{V_w}{V_3} \Rightarrow V_0 = V_3 (1-f) \]

\[ W = \frac{P_0 (V_3 (1-f))^k}{-k+1} \left( V_3^{-k+1} - (V_3 (1-f))^{-k+1} \right) \]

\[ = \frac{P_0 V_3^k (1-f)^k}{-k+1} \left( V_3^{-k+1} - V_3^{-k+1} (1-f)^{-k+1} \right) \]

\[ = \frac{P_0 V_3^k V_3^{-k+1}}{-k+1} \left( (1-f)^k - (1-f)^k (1-f)^{-k+1} \right) \]

NOTES

Adiabatic expansion is assumed and isentropic process. Air is doing work on a water volume. That work is determined by the pressure & volume of the air.
\[ W = \frac{P_0 V_s}{-K+1} \left[ (1-f)^K - (1-f) \right] \]

* Convert to work per unit mass

\[ \frac{W}{m} = \frac{\left[ \frac{P_0 V_s}{-K+1} \left[ (1-f)^K - (1-f) \right] \right]}{m} \]

\[ \omega = \frac{\frac{P_0 V_s}{-K+1} \left[ (1-f)^K - (1-f) \right]}{m_s + (P_w + V_T)} \]

**CHECK CALCULATION**

\[ \omega = \left[ \frac{(41 \text{ psi})(6894.76 \text{ Pa}) + (100 \times 10^3 \text{ Pa})}{(-1.4 + 1)(1/1081 \text{ mL})(1/100 \text{ mL})} \left( (1-0.35)^{1.4} - (1-0.35) \right) \right] \]

\[ = \left( \frac{(181.9 \times 10^3 \text{ Pa}) + (1000 \text{ kg/m}^3)(1.35)(12.36 \text{ mL})(1 \text{ mL}/1 \text{ kg})}{-1084.206245 \times 10^3 \text{ J}) (-0.1028884962) \right) \]

\[ = \frac{542.4}{1960} \text{ J/kg} \]

**NOTES**

The mass changes with volume of water. With more volume, there is also more mass to lift, affecting the optimum volume.

This optimum volume is looked up by looking up the maximum performance from 0 - 1 filling factor.
HEIGHT PREDICTION OVERVIEW

1) Phase I Thrust (Water Thrust)
\[ T_n = \frac{dm}{dt} V_\text{inlet} \]
\[ \frac{dm}{dt} = P_w A_n V_\text{inlet} C_{L2} \]
\[ T_n = P_w A_n V_\text{inlet}^2 C_{L2} \]

* Use Bernoulli's Equation (Incompressible Flow)
\[ \frac{P}{P_w} + \frac{V_\text{inlet}^2}{2} + \alpha z_1 = \frac{P_{\text{atm}}}{P_w} + \frac{V_{\text{exit}}^2}{2} + \alpha z_2 \]
\[ V_{\text{exit}} = \sqrt{2 \left( \frac{P - P_{\text{atm}}}{P_w} \right) + \alpha z_2} \]

Phase 1 Thrust (Water)

Volume of Air

\[ \sum F_y = m(t) = T_n - F_D - W \]
\[ a = \frac{T_n - F_D}{m(t)} - g \]
\[ u = \int_{t_0}^{t} a \, dt \]
\[ h = \int_{t_0}^{t} u \, dt \]

2) Phase II Thrust (Air Burst)

3) Phase III (No Thrust)

4) F_Drag

5) Velocity, Height, Impulse

NOTES
- Air is assumed ideal gas
- Adiabatic/isentropic expansion
- Nozzle coefficient is assumed to be necessary

The only variable in the thrust equation is the velocity of the exiting fluid (water). So that must be solved for. The exhaust velocity is solved for in terms of pressure using Bernoulli's equation.

Pressure could not be calculated over time, but volume could.
\[ \frac{V}{\\rho V} = \sqrt{\frac{2(P_0(v_0/\gamma)^k - P_{\text{atm}})}{\rho W}} + \alpha T Z_i \]

* EQUATION FOR VOLUME MUST BE DERIVED, OVER TIME

\[ \frac{dm}{dt} = \rho w A_n V V_{\text{ave}} \Rightarrow \frac{dV}{dt} = A_n V V_{\text{ave}} \]

\[ \frac{dV}{dt} = C_{\text{ave}} A_n \sqrt{\frac{2(P_0(v_0/\gamma)^k - P_{\text{atm}})}{\rho W}} \]

Nozzle efficiency can be added to take it into account,

\[ \frac{dV}{dt} = C_{\text{ave}} A_n \sqrt{\frac{2(P_0(v_0/\gamma)^k - P_{\text{atm}})}{\rho W}} \]

\[ \int_0^t \frac{dV}{dt} \, dt = V(t + \Delta t) \approx V(t) + \Delta t \left( \frac{dV}{dt} \right) \]

\[ V(t + \Delta t) \approx V(t) + \Delta t C_{\text{ave}} A_n \sqrt{\frac{2(P_0(v_0/\gamma)^k - P_{\text{atm}})}{\rho W}} \]

Valid if \( V < V_{\text{max}} \)

CHECK CASES

Initial \( V = V_{\text{init}} - A W = 0.101081 \, \text{m}^3 - 1.000 \times 10^{-4} \, \text{m}^3 = 1000 \times 10^{-4} \, \text{m}^3 \)

\[ V(t + \Delta t) = 1000 \times 10^{-4} \, \text{m}^3 + (1.005)(1) \left( \frac{2.75159}{1000} \right) \]

\[ \sqrt{\frac{2(5826085 \, \text{Pa})(0.101081/1000 \times 10^{-4} \, \text{m}^3)^{1/4} - (1000 \times 10^{-4} \, \text{Pa})}}{1000 \, \text{Kg/m}^3} \]

\[ = 1000 \times 10^{-4} \, \text{m}^3 + (3.66 \times 10^{-4} \, \text{m}^2)(1.005)(23.779 \, \text{m/s}) = 40.89 \times 10^{-4} \, \text{m}^3 \]

NOTES

B/C THE INTEGRAL OF THE DERIVATIVE EQUATION IS DEEPLY EMBEDDED INTO THE EQUATION. A NUMERICAL INTEGRATION IS REQUIRED TO INTEGRATE THE DERIVATIVE. TO START THE NUMERICAL INTEGRATION, THE INITIAL VOLUME OF AIR WITH THE ROCKET CHAMBER IS USED.
THRUST

\[ T_h = P_w A_n \frac{V_0}{\rho} \sqrt{2 \left( \frac{P_0 (V_0/V)^k - P_{\text{atm}}}{\rho_0} \right) + C_t Z_1} \]

\[ = A_n C_m L \left( 2 \left( \frac{P_0 (V_0/V)^k - P_{\text{atm}}}{\rho_0} \right) + \frac{P_w A_T Z_1 A_n L C_m}{\text{THRUST}} \right) \]

\[ \Delta T = \frac{\rho_0}{C_{\text{m}}, \text{excess}} \cdot 0.9 \left( \frac{\nu_0}{V_{\text{tot}}} \right) L \]

\[ Z_1 = 0.9 \left( \frac{\nu_0}{V_{\text{tot}}} \right) L \]

CHECK CASE 4

\[ T_h = \left( V_4 \right) \left( \frac{1}{2} \left( 38 \text{ m/s} \right)^2 \right) (1) \left( \begin{array}{c} \left( 6 \text{ kg/m}^3 \right) \left( \frac{V_4 \text{ m}}{1.0259 \text{ m}^3} \right) \left( 9.81 \text{ m/s}^2 \right) \left( 34.75 \text{ g/s} \left( 9.807 \text{ m/s}^2 \right) \right) \right) * \]

\[ \left( 0.9 \left( \frac{1000 \text{ kg/m}^3}{1000 \text{ g/m}^3} \right) \left( 1.191 \text{ m/s}^2 \right) \right) \]

\[ = 2.00 \times 10^3 \text{ N} + 3133 \text{ N} \]

\[ = 2410 \text{ N} \]

\[ T_h = 2.41 \times 10^3 \text{ N} \]

NOTES:

The potential energy of the water is taken back into account at thrust 8/10. It can be approximated by dividing the total potential energy in the system by the number of time increments it takes for all the water to expel, multiply it by a given acceleration, and distribute it across thrust phase 1. The easiest way to do this in the simulation was to use the initial thrust.

0.9 is an approximation to account for soda bottle necking. It was derived by taking the average of various factors determined from exp. measuring the water height in a variety of bottles at a variety of volumes.
MASS \( m(t) \)

\[
m(t) = m_0 + Ma + Pw (V_{tot} - V)
\]

\[
Ma = \frac{P_0 V_0}{R \omega T_0}
\]

\[
m(t) = m_0 + \frac{P_0 V_0}{R T_0} + Pw (V_r - V)
\]

CHECK CASE

\[
m(t) = 1.1819 \text{ kg} + \frac{(382,085 \text{ Pa})(100 \text{ m}^3)}{(288 \text{ J/kg K})(200 + 293 \text{ K})} + (1000 \text{ kg/m}^3)(100 \text{ m}^3 - 100 \text{ m}^3)
\]

\[
= 1.1819 \text{ kg} + 1004902381 \text{ kg} + 0.4 \text{ kg} = 0.587 \text{ kg}
\]

IDEAL GAS LAW IS USED. To is used for temp of air b/c ADIABATIC expansion is assumed for phase 1 of thrust.
PHASE II

PHASE II THRUST

\[ \frac{P_1}{P_2} = \left[1 + \frac{(K-1)}{2}\left(\frac{V_{Doe}}{C}\right)^2 \right]^{\frac{K}{K-1}} \]

\[ C = \sqrt{\frac{V_{Doe}}{KRT}} \]

\[ V_{Doe} = \left[ \left(\frac{P_0}{P_0(\frac{V_0}{V_0})^{\frac{K-1}{2}}} - 1 \right) \left(\frac{2}{K-1}\right) \right] \sqrt{KRT \left(\frac{V_0}{V_0} - \frac{1}{2}\right)} \]

\[ \frac{dV}{dt} = C_{Moe} A_n \sqrt{\frac{V}{KRT_0 \left(\frac{V_0}{V_0} - \frac{1}{2}\right)} \left[ \left(\frac{P_0(\frac{V_0}{V_0})^{\frac{K-1}{2}}} {P_{atm}}\right) - 1 \right] \left(\frac{2}{K-1}\right)} \]

\[ V(t+\Delta t) \approx V(t) + \Delta t \left[ C_{Moe} A_n \sqrt{\frac{V_0}{KRT_0 \left(\frac{V_0}{V_0} - \frac{1}{2}\right)} \left[ \left(\frac{P_0(\frac{V_0}{V_0})^{\frac{K-1}{2}}} {P_{atm}}\right) - 1 \right] \left(\frac{2}{K-1}\right)} \right] \]

NOTES

\[ \text{B/C THE EXCESS AIR OF THRUST PHASE II IS COMPRESSIBLE, UNLIKE LIQUIDS, THE PRESSURE IN THE BERNOUlli EQUATION IS DIFFERENTIAL. THE SIMPLIFIED EQUATION IS SEEN ABOVE; A DERIVATION CAN BE FOUND IN [COURTEL, TURNER, (2005), FUNDAMENTALS OF THERMAL-FLUID SCIENCES, "PG. 540,].} \]

\[ \frac{V_{Doe}}{C} = M_0 \left[ \text{Mach #} \right] \]

\[ \text{THE CHANGING TEMPERATURE OF THE AIR IS ALSO TAKEN INTO ACCOUNT IN THIS PHASE.} \]
\[ V(t+\Delta t) = 1001085 \text{ m}^3 + (10010) \left[ 1 \left( \frac{1}{1.4} \left( \frac{2.88 \text{ J/kg.K}}{293 \text{ K}} \right) \left( \frac{1001085 \text{ m}^3}{10000 \text{ m}^3} \right)^{-0.41} \right) \right] \]

\[ \sqrt{\left( \frac{3.82 \times 10^5 \text{ Pa}}{100 \times 10^3 \text{ Pa}} \right) \left( \frac{1001085 \text{ m}^3}{10000 \text{ m}^3} \right)^{1.4-1} \left[ \frac{2}{1.4-1} \right]} \]

\[ = 1001085 \text{ m}^3 + (10010) \left( 3.16 \times 10^9 \times 10^{-4} \text{ m}^2 \right) \left( 318 \times 10^3 \text{ m/s} \right) \left( 1043800 \text{ N} \right) \]

\[ = 1001205 \text{ m}^3 \]

**THRUST**

\[ T_h = \text{Par} \times \text{An} \times \frac{V_o^2}{c_{w2}} \]

\[ = \text{Par} \times \text{An} \times \left( \left[ \frac{P_0 \left( \frac{V_0}{V} \right)^K}{P_{atm}} \right]^{\frac{K-1}{K}} \left[ \frac{2}{K-1} \right] \right)^2 \left( \frac{KRT_0 \left( \frac{V}{V_0} \right)^{-0.4}}{P_{atm}} \right) \]

\[ = \text{Par} \times \text{An} \times \left( \frac{KRT_0 \left( \frac{V}{V_0} \right)^{-0.4}}{P_{atm}} \right) \left[ \left( \frac{P_0 \left( \frac{V_0}{V} \right)^K}{P_{atm}} \right)^{\frac{K-1}{K}} \left[ \frac{2}{K-1} \right] \right]^2 \]

\[ = \text{Par} \times \text{An} \times \left( KRT_0 \left( \frac{V}{V_0} \right)^{-0.4} \right) \left[ \left( \frac{P_0 \left( \frac{V_0}{V} \right)^K}{P_{atm}} \right)^{\frac{K-1}{K}} \left[ \frac{2}{K-1} \right] \right] \]

**CHECK CALCULATIONS**

\[ T_h = (1.2 \text{ kg/m}^3) \left( \frac{1}{1.0159 \text{ m}^3} \right) \left( 1.9 \times 2.88 \text{ J/kg.K} \times 293 \text{ K} \left( \frac{1001085}{10000 \text{ m}^3} \right)^{-0.4} \right) \times \]

\[ \left[ \left( \frac{3.82 \times 10^5 \text{ Pa}}{100 \times 10^3 \text{ Pa}} \right) \left( \frac{1001085 \text{ m}^3}{10000 \text{ m}^3} \right)^{1.4-1} \left[ \frac{2}{1.4-1} \right] \right] \]

\[ = 410.933 \text{ N} \]
MASS

Mass of exhausting air = $\rho \left( \frac{dV}{dt} \right) dt$

$m(t+\Delta t) = m(t) - \rho \left( \frac{dV}{dt} \right) \Delta t$

Check Calculation:

\[
\frac{m(t+\Delta t)}{m(t)} = (1.1856 \text{ kg}) - (1.2 \text{ kg/m}^3) \left( \frac{1000}{1.02159 \text{ m}^3} \right) \left( \frac{\pi}{4} \right) \left( \frac{1021.59 \text{ m}^2}{10000 \text{ m}^2} \right)^{1/4} \left( \frac{288 \text{ J/kg} \cdot \text{K}}{293 \text{ K}} \right) \left( \frac{1 \text{ Pa}}{1 \text{ Pa}} \right) \left( \frac{1000 \text{ m}^3}{1000 \text{ m}^3} \right)^{1/4} \left( \frac{1.4-1}{1.4} \right) \left( \frac{1196 \text{ m}^3/s}{1.1856 \text{ kg}} \right)
\]

\[
= \frac{1.1856 \text{ kg} - (1.2 \text{ kg/m}^3) (1.19 \text{ m}^3/s) \left( \frac{1.1856 \text{ kg}}{1.196 \text{ m}^3/s} \right)}{1.1856 \text{ kg}} = 0.9955 \text{ kg}
\]

Notes:

Just as the mass of the water was determined based on instantaneous volume in Phase 1, the mass of the air is determined by using the instantaneous volume. Except, the volume flow rate is used to calculate how much mass of air is leaving at the instantaneous time. This mass is subtracted from the previous mass.
**Phase III - No Thrust**

\[ \nu_{air} = NA \]
\[ T_h = 0 \]
\[ m(t) = \text{constant} = m_0 \]

**Formula**

\[ F_0 = \frac{1}{2} C_D \rho_{air} A B U^2 = \frac{C_{D0} \rho_{air} A_B U \nu_{U} U}{2} \]
\[ F_0' = F_0(t + \Delta t) = \frac{C_{D0} \rho_{air} A_B U(t + \Delta t) U(t)}{2} \]

\[ \Delta U(t + \Delta t) = U(t) + \Delta t \left( \frac{T_h - F_0'}{m(t)} - g \right) \]

\[ F_0 = \frac{C_{D0} \rho_{air} A_B U \nu_{U} U'}{2} \]

**Note:**

The volume of air within the rocket chamber is at atmospheric pressure and therefore has no effect on rocket coast. The rocket is a projectile in this phase, listed in the simulator is the volume of excess air at atmospheric pressure.

**Drag Force**

\[ F_0 = (13)(1.26 kg/m^3)(\frac{A}{t}(109398 m^2))(1.341328 m/s)) \]
\[ = 8000145 N \]

**Notes:**

The drag force requires a velocity so a separate table is made by a drag calculation referencing a previous velocity that refers to the instantaneous drag force. Then the drag calculation used to predict actual velocity referenced this velocity in the second table.
\[ a = \frac{T_h - F_d}{m(t)} - g \]

* Take into account launch angle

\[ a = \frac{(\sin \theta) T_h - (\sin \theta) F_d}{m(t)} - g \]

\[ \frac{\sin \theta (T_h - F_d)}{m(t)} - g \]

\[ u = \int \frac{du}{dt} \, dt = u(t+\Delta t) = u(t) + \Delta t \left[ \frac{\sin \theta (T_h - F_d)}{m(t)} - g \right] \]

**Check cases**

\[ u(t+\Delta t) = 0 + (1.0 \times 9.882 \text{ m/s}^2) \left[ \frac{\sin 45^\circ (2,823.353 \text{ N} - 1,800.195 \text{ N})}{0.57282 \text{ kg}} - 9.882 \text{ m/s}^2 \right] \]

\[ = 341.3 \text{ m/s} \]
\[ h = \int_{t=0}^{t} \frac{\partial h}{\partial t} dt = h(t) + \frac{\Delta h}{\Delta t} (u(t) + \frac{1}{2} (u(t) + u(t+\Delta t))) + u(t+\Delta t) \]

**Check Cases**

\[ h = (0) + \frac{(0.0015)}{(0} + 4 \left( \frac{0 + 1.513 m/s}{2} \right) + 1.3413 m/s \]

\[ = 10.4071 m \]

**Impulse**

\[ I = m_2 u_2 - m_1 u_1 \]

\[ \sum I = I(0) + \left[ m(u(t+\Delta t)) - m(0) \right] \]

**Check Cases**

\[ \sum I = (0) + \left[ (1570 kg)(1.3413 m/s) - (1584 kg)(0) \right] \]

\[ = 0.197 N\cdot s \]

**Notes**

A SIMPSON NUMERICAL INTEGRATION IS USED FOR THE IMPULSE SINCE INITIALLY, A 1ST & 2ND VALUES FOR VELOCITY IS AVAILABLE & TO GET SIMPSON INTEGRATION IS MORE ACCURATE. ONLY A 1ST VALUE WAS AVAILABLE INITIALLY, SO A BASIC INTEGRATION HAD TO BE USED.
\[
1 - \frac{h_{\text{meas}}}{h_{\text{exp}}} \times 100 = \% \text{ ERROR}
\]

**Check Calculation**

\[
\% \text{ ERROR} = 1 - \frac{19.437 \text{ m}}{19.877 \text{ m}} (1.010) = 2.18 \%
\]

**Statistical Data**

\[
h_{\text{exp}} = \frac{\sum h_{\text{exp}}}{n} = \frac{(19.87 + 19.2 + 19.8 + 20.72 + 19.51 + 19.8 + 20.11 + 19.72 + 19.91 + 19.9\text{ cm})}{10} = 19.81 \text{ m}
\]

\[
s = \sqrt{\frac{\sum (h_{\text{exp}} - h_{\text{exp}})^2}{n - 1}} = \sqrt{\frac{(19.87 - 19.81)^2 + (19.72 - 19.81)^2 \ldots}{10 - 1}} = 0.42 \text{ m}
\]

\[
s_\bar{x} = \sqrt{\frac{s^2}{n}} = \sqrt{\frac{(0.42 \text{ m})^2}{10}} = 0.13 \text{ m}
\]

\[
\text{CI (±)} = (t)(s_\bar{x})
\]

\[
\text{CI (±) }_{\% \text{ error}} = (2.228)(0.545 \%) = 1.2192\% \text{ error}
\]

**Notes**

In the data analysis cell, the logic function is built into the cell to accommodate text values if a number is in the cell. \( t \) is the two-tailed t-score value found in standard statistical tables. They are calculated by the degrees of freedom & the desired confidence. A 95% confidence is common.

\( CI(\pm) \) is the confidence interval w/o the mean.
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<td>Filling Factor [Water Volume/Total Volume]</td>
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</table>
\[ t = \text{Time} \]
\[ T_{th} = \text{Water Rocket Thrust at Phase n of Thrust} \]
\[ F_D = \text{Drag Force} \]
\[ W = \text{Weight of Rocket} \]
\[ m_n = \text{Mass of Rocket at Phase n Thrust} \]
\[ m_a = \text{Mass of Excess Air} \]
\[ m_b = \text{Mass of Empty Rocket with Ballast Mass} \]
\[ m_e = \text{Mass of Empty Rocket Without Ballast Mass} \]
\[ m_b = \text{Mass of Ballast} \]
\[ m_{ho} = \text{Optimized Mass of Ballast} \]
\[ a_n = \text{Acceleration at Phase n Thrust} \]
\[ C_D = \text{Drag Coefficient} \]
\[ \rho_{air} = \text{Density of air} \]
\[ u_n = \text{Velocity at Phase n Thrust} \]
\[ A = \text{Cross Sectional Area of Rocket} \]
\[ A_n = \text{Cross Sectional Area of Nozzle} \]
\[ v_D = \text{Exit Velocity of Flow at Nozzle} \]
\[ \frac{dm_p}{dt} = \text{Mass Flow Rate of Exiting Fluid at Nozzle} \]
\[ \rho_{fuel} = \text{Density of Fuel} \]
\[ P_0 = \text{Initial Pressure in Water Chamber} \]
\[ P = \text{Instantaneous Pressure in Adiabatic Expansion} \]
\[ P_{atm} = \text{Atmospheric Pressure} \]
\[ V_0 = \text{Initial Volume of Air in Water Chamber} \]
\[ V_{tot} = \text{Total Volume of Water Chamber} \]
\[ V_w = \text{Water Volume in Water Chamber} \]
\[ V = \text{Instantaneous Volume of Air in Adiabatic Expansion} \]
\[ V_n = \text{Volume of Air in Chamber at Phase n Thrust} \]
\[ k = \text{Ratio of Specific Heats of Air} \]
\[ \frac{dV}{dt} = \text{Volume Flow Rate of Air} \]
\[ c = \text{Speed of Sound for Air} \]
\[ T = \text{Instantaneous Temperature of Air in Water Chamber} \]
\[ T_0 = \text{Initial Temperature of Air in Chamber} \]
\[ R = \text{Gas Constant for Air} \]
\[ h = \text{Instantaneous Height of Rocket} \]
\[ I = \text{Instantaneous Impulse of Rocket} \]
\[ C_p = \text{Center of Pressure of the Rocket} \]
\[ C_m = \text{Center of Mass of the Rocket} \]
\[ L_e = \text{Length of Entire Rocket} \]
\[ L_b = \text{Length of Ballast} \]
\[ r_{in} = \text{Inner Radius of Water Chamber} \]
\[ r_{out} = \text{Outer Radius of Water Chamber} \]
\[ \sigma_r = \text{Radial Stress} \]
\[ \sigma_n = \text{Hoop Stress (Circumferential)} \]
\[ \sigma_a = \text{Axial Stress (Longitudinal)} \]
\[ \sigma_y = \text{Yield Strength in Tension} \]
\[ \sigma_{UTS} = \text{Ultimate Tensile Strength} \]
\[ t_{wall} = \text{Wall Thickness of Water Chamber} \]
\[ E = \text{Modulus of Elasticity of Water Chamber Material} \]
\[ V_{act} = \text{Actual Total Volume Due to Volumetric Strain} \]
\[ w = \text{Work/Unit Mass} \]
\[ f = \text{Filling Factor} \]
\[ P_g = \text{Gauge Pressure} \]
Rocket Flight Results Equations [Step 4]

\[ \Sigma F_y = T_h - F_D - W = ma. \]  \hspace{1cm} (1)

\[ F_D = \frac{1}{2} C_D \rho_{air} u^2 A. \]  \hspace{1cm} (2)

\[ T_h = v_{D/e} \frac{dm_e}{dt}. \]  \hspace{1cm} (3)

\[ \frac{dm_e}{dt} = \rho_{fluid} A_n v_{D/e}. \]  \hspace{1cm} (4)

\[ T_h = \rho_{fluid} A_n v_{D/e}^2. \]  \hspace{1cm} (5)

\[ \frac{P_1}{\rho_{fluid}} + \frac{v_1^2}{2} + gz_1 = \frac{P_2}{\rho_{fluid}} + \frac{v_2^2}{2} + gz_2. \]  \hspace{1cm} (6)

\[ \frac{(P-P_{atm})}{\rho_{fluid}} = \frac{v_{D/e}^2}{2}. \]  \hspace{1cm} (7)

\[ P = P_0 (V_0/V)^k. \]  \hspace{1cm} (8)

\[ \frac{dV}{dt} = C_{noz} A_n v_{D/e}. \]  \hspace{1cm} (9)

\[ \frac{dV}{dt} = C_{noz} A_n \sqrt{\frac{2[P_0(V_0/V)^k-P_{atm}]}{\rho_w}}. \]  \hspace{1cm} (10)

\[ V_1(t + \Delta t) \approx V(t) + \Delta t C_{noz} A_n \sqrt{\frac{2[P_0(V_0/V)^k-P_{atm}]}{\rho_w}}. \]  \hspace{1cm} (11)

\[ \frac{P_1}{P_{atm}} = \left[ 1 + (k - 1/2) \left( \frac{v_{D/e}}{c} \right) \right]^{\frac{k}{k-1}}. \]  \hspace{1cm} (12)

\[ c = \sqrt{kRT}. \]  \hspace{1cm} (13)

\[ \frac{dV}{dt} = C_{noz} A_n c \sqrt{\left( \frac{P_0(V_0/V)^k}{P_{atm}} \right)^{\frac{k-1}{k}} - 1} \left[ \frac{2}{k-1} \right]. \]  \hspace{1cm} (14)

\[ V_2(t + \Delta t) \approx V(t) + \Delta t C_{noz} A_n c \sqrt{\left( \frac{P_0(V_0/V)^k}{P_{atm}} \right)^{\frac{k-1}{k}} - 1} \left[ \frac{2}{k-1} \right]. \]  \hspace{1cm} (15)

\[ T = T_0 (V/V_0)^{-0.4}. \]  \hspace{1cm} (16)

\[ m_1(t) = m_s + m_a + \rho_w (V_{tot} - V), \ for \ V < V_{tot}. \]  \hspace{1cm} (17)
\[ m_a = \frac{p_0 V_0}{RT_0}. \] (18)

\[ m_2(t + \Delta t) = m(t) - \left(\frac{dv}{dt}\right)_t \rho_{\text{air}} \Delta t, \text{ for } V_{\text{tot}} < V < V_{\text{atm}}. \] (19)

\[ a_1 = \frac{2A_n\rho_{\text{air}}C_{\text{No2}} \left[p_0(V_0/V)^k - P_{\text{atm}}\right] + \frac{g \rho_w g l_t V_w}{t_{\text{thrust}}/\Delta t}}{m_s + \frac{p_0 V_0}{RT_0} + \rho_w (V_{\text{tot}} - V)} - g, \text{ for } V < V_{\text{tot}}. \] (20)

\[ a_2 = \frac{\rho_{\text{air}} A_n C_{\text{No2}} \left[p_0(V_0/V)^k \right]^{k-1} \left(\frac{2}{k-1}\right) \frac{1}{2} \rho_{\text{air}} A|u|u}{m_s + \frac{p_0 V_0}{RT_0}} - g, \text{ for } V_{\text{tot}} < V < V_{\text{atm}}. \] (21)

\[ a_3 = \frac{-1}{2} \rho_{\text{air}} A|u|u - g, \text{ for } V = V_{\text{atm}}. \] (22)

\[ u_1(t + \Delta t) \approx u(t) + \Delta t[a_1], \text{ for } V < V_{\text{tot}}. \] (23)

\[ u_2(t + \Delta t) \approx u(t) + \Delta t[a_2], \text{ for } V_{\text{tot}} < V < V_{\text{atm}}. \] (24)

\[ u_3(t + \Delta t) \approx u(t) + \Delta t[a_3], \text{ for } V = V_{\text{atm}}. \] (25)

\[ h(t + \Delta t) \approx h(t) + \frac{\Delta t}{6} \left[u(t) + 4 \left(\frac{u(t) + u(t + \Delta t)}{2}\right) + u(t + \Delta t)\right]. \] (26)

\[ l = l(t) + [m(t + \Delta t)u(t + \Delta t) - m(t)u(t)], \] (27)

\[ t(t + \Delta t) = t + \Delta t. \] (28)

\[ T_{h1} = 2A_n\rho_{\text{air}}C_{\text{No2}} \left[p_0(V_0/V)^k - P_{\text{atm}}\right] + \frac{g \rho_w (g + a_{\text{avg,rocket}}) l_t V_w}{t_{\text{thrust}}/\Delta t}, \text{ for } V < V_{\text{tot}}. \] (29)

\[ T_{h2} = \rho_{\text{air}} A_n C_{\text{No2}} C^2 \left[p_0(V_0/V)^k \right]^{k-1} \left(\frac{2}{k-1}\right) \frac{1}{2}, \text{ for } V_{\text{tot}} < V < V_{\text{atm}}. \] (30)

\[ u(t + \Delta t) \approx u(t) + \Delta t \left(\frac{\sin\theta \left(T_{h1} - F_D\right)}{m(t)} - g\right). \] (31)

Ballast Mass Equations [Step 2]

\[ C_p = 0.5 \times L_t. \] (32)

\[ C_m = \frac{0.25 L_t m_e + (L_t - 66 L_b) m_b}{m_e + m_b} \] (33)

IF \( C_m > 1.125 C_p \), ‘GO’, ELSE ‘NO GO’ (34)
\[ m_{bo} = \frac{0.3125L_t m_e}{A_{375L_t - 0.66L_b}} \]  

(35)

**Initial Pressure Equations [Step 3]**

**IF** \( t_{wall} \leq 0.1r_{in} \), ‘THIN WALL’, ELSE ‘THICK WALL’

\[ \sigma_r = P_g \]  

(36)

\[ \sigma_h \text{ Thin Wall } = P_g \left( \frac{r_{in} + r_{out}/2}{t_{wall}} \right) \]  

(37)

\[ \sigma_h \text{ Thick Wall } = P_g \left( \frac{r_{in}^2 + r_{out}^2}{r_{out}^2 - r_{in}^2} \right) \]  

(38)

\[ \sigma_a \text{ Thin Wall } = P_g \left( \frac{r_{in} + r_{out}/2}{2t_{wall}} \right) \]  

(39)

\[ \sigma_a \text{ Thick Wall } = P_g \left( \frac{r_{in}^2}{r_{out}^2 - r_{in}^2} \right) \]  

(40)

\[ V_{act} = \left( \frac{\sigma_h \sigma_a}{E^2} + \frac{\sigma_h^2 + 2\sigma_h \sigma_a}{E^2} + \frac{2\sigma_h + \sigma_a}{E} \right) V_0 + V_0 \]  

(41)

\[ P_0 \text{ max for elastic region } = 0.95t_{wall} \sigma_y / \left( \frac{r_{in} + r_{out}}{2} \right) \text{ or } 0.95 \sigma_y \]  

(42)

\[ P_0 \text{ max for UTS } = 0.95 \sigma_{UTS} / \left( \frac{r_{in}^2 + r_{out}^2}{r_{out}^2 - r_{in}^2} \right) \text{ or } 0.95 \sigma_{UTS} \]  

(43)

**Flight Parameter Equations [Step 4]**

\[ w = \left( \frac{1}{m_s + (\rho_w) V) \right) \left( \frac{V_{tot}}{V_0} \right)\left( (1 - f)^k - (1 - f) \right) \]  

(44)

\[ f = \frac{V_w}{V_{tot}} \]  

(45)